

CURVED A_∞ -ALGEBRAS AND GAUGE THEORY

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ABSTRACT. We propose a general notion of algebraic gauge theory obtained via extracting the main properties of classical gauge theory. Building on a recent work on transferring curved A_∞ -structures we show that, under certain technical conditions, algebraic gauge theories can be transferred along chain contractions. Specializing to the case of the contraction from differential forms to cochains, we obtain a simplicial gauge theory on the matrix-valued simplicial cochains of a triangulated manifold. In particular, one obtains discrete notions of connection, curvature, gauge transformation and gauge invariant action.

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1. INTRODUCTION

Lattice gauge theory has been a widely used tool in computational physics for several decades and has also been generalized in various directions in the mathematical literature, e.g. to general simplicial complexes [12]. It is however of considerable interest to develop alternative discretization schemes in which all fundamental differential geometric and algebraic structures present in classical gauge theory have direct and transparent analogues. In this paper we propose a general algebraic framework for gauge theory and apply it to obtain a simplicial gauge theory on simplicial cochains, based on the notion of curved A_∞ -algebra.

We recall that A_∞ -algebras [14] are a generalization of differential graded (dg) algebras in which the associativity condition is replaced by an infinite sequence of identities involving higher “multiplications”. Curved A_∞ -algebras (cf. [6] and [10]) are a natural generalization of both A_∞ -algebras and curved dg algebras [13]. The latter are generalizations of dg algebras which extract the algebraic properties of connections and curvatures on vector bundles.

It was recently shown in [11] that under certain technical conditions one can transfer along chain contractions curved dg structures to curved A_∞ -structures, building on previous transfer results (see e.g. [7] and [8]). Applying this to Dupont’s contraction from differential forms to cochains [4], one obtains transferred notions of connection and curvature on the matrix-valued simplicial cochains of a triangulated manifold. The main theme of the present paper is transferring the remaining ingredients of classical gauge theory, namely gauge transformations, inner product

and gauge-invariant action, to simplicial cochains. In order to approach this problem, we introduce a general notion of *algebraic gauge theory* which we explain in more detail below.

In Section 2 we present background material on curved A_∞ -algebras and their morphisms and discuss gauge transformations in dg algebras. In Section 3 we define algebraic gauge theory in the particular case of curved dg structures and provide three main examples: classical gauge theory on a smooth manifold, non-commutative gauge theories arising from A. Connes' spectral triples and gauge theories defined by deformations of dg algebras. Section 4 is devoted to certain properties of special chain contractions and auxiliary results related to the homological perturbation lemma that are needed in the sequel.

In Section 5 we define an algebraic gauge theory over an arbitrary groupoid as a functor from the groupoid to a category whose objects are triples consisting of a graded vector space V , a preferred class of curved A_∞ -structures on V and an inner product on the tensor coalgebra of V , and whose morphisms are tensor coalgebra maps preserving the additional structure. We then prove our main result, Theorem 5.4, which can be formulated as follows: the algebraic gauge theory given by the deformations of a fixed dg algebra can be transferred along a special chain contraction provided that its homotopy commute with the gauge group action. Moreover, there exists a natural transformation between the initial and the transferred functor. Applying this transfer result to Dupont's contraction we obtain, in the case of a trivial bundle, a simplicial gauge theory on the matrix-valued simplicial cochains of a triangulated manifold.

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2. CURVED A_∞ -ALGEBRAS AND THEIR MORPHISMS

We work over a fixed ground field \mathbf{k} of characteristic 0; all chain complexes have differentials of degree +1. In what follows $V = \bigoplus_p V_p$ always stands for a graded vector space over \mathbf{k} , its *suspension* sV is defined by $(sV)_p = V_{p+1}$. We write $V^{\otimes k}$ for the k -th graded tensor power of V so that $V^{\otimes 0} = \mathbf{k}$. We denote by

$$T(sV) = \mathbf{k} \oplus sV \oplus sV^{\otimes 2} \oplus \dots$$

the full tensor coalgebra over the shifted space sV . We always consider $T(sV)$ as a graded vector space with respect to the total grading induced by the grading on V and the grading given by the number of factors in the tensor product. When applying graded maps to graded objects, we always use the *Koszul sign rule*, by which we mean the appearance of a sign $(-1)^{\deg(a)\deg(b)}$ when switching two adjacent graded symbols a and b .

Definition 2.1. A *curved A_∞ -structure* on a graded vector space V is a degree 1 graded coderivation D on $T(sV)$ such that $D^2 = 0$. The pair (V, D) is called a *curved A_∞ -algebra*.

The following proposition provides a useful explicit description of curved A_∞ -structures.

Proposition 2.2. (cf. [6]) *There is a bijection between curved A_∞ -structures on V and collections of linear maps $\{m_k\}_{k=0}^\infty$ such that*

$$m_k : V^{\otimes k} \rightarrow V$$

is of degree $2 - k$, satisfying for every $n \geq 0$ the identity

$$(2.1) \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1} (\mathbf{1}_V^{\otimes r} \otimes m_s \otimes \mathbf{1}_V^{\otimes t}) = 0.$$

The relationship between a curved A_∞ -structure D and the maps $\{m_k\}$ is obtained as follows. Setting

$$D_{n,k} := pr_{(sV)^{\otimes k}} \circ D|_{(sV)^{\otimes n}} : (sV)^{\otimes n} \rightarrow (sV)^{\otimes k},$$

one has

$$D_{n,k} = \sum_{\substack{r+s+t=n \\ r+1+t=k}} \mathbf{1}_V^{\otimes r} \otimes D_{s,1} \otimes \mathbf{1}_V^{\otimes t},$$

where $D_{k,1}$ coincides with m_k up to a sign (see [6, Lemma 1.3] for details).

We shall refer to $m_0(1) = D1$ (where 1 denotes the unit of \mathbf{k}) as the *curvature* and to m_1 as the *connection* (or covariant derivative) of a given curved A_∞ -structure.

The notion of curved A_∞ -algebra is a generalization of the notions of A_∞ -algebra introduced by Stasheff in [14] and of curved differential graded (dg) algebra introduced by Positsel'skii in [13]. More specifically, an A_∞ -algebra structure is given by a collection of maps $\{m_k\}_{k=0}^\infty$ as in Proposition 2.2 such that $m_0 = 0$ and a curved dg algebra structure is given by a collection of maps such that $m_k = 0$ for $k > 2$. Thus, a curved dg structure is defined by the first four identities in (2.1) which in this case read

$$(2.2) \quad m_1 m_0 = 0,$$

$$(2.3) \quad m_1 m_1 = m_2(m_0 \otimes \mathbf{1}) - m_2(\mathbf{1} \otimes m_0),$$

$$(2.4) \quad m_1 m_2 = m_2(m_1 \otimes \mathbf{1}) + m_2(\mathbf{1} \otimes m_1),$$

$$(2.5) \quad m_2(m_2 \otimes \mathbf{1}) = m_2(\mathbf{1} \otimes m_2).$$

We note that (2.2) can be interpreted as an abstract Bianchi identity, (2.3) says that square of the connection m_1 equals a commutator with the curvature, (2.4) expresses the fact that m_1 is a (graded) derivation and the last identity amounts to the associativity of multiplication given by m_2 . Examples of curved dg algebras arising from vector bundles and projective modules will be described in Section 3.

Definition 2.3. A *morphism* between two curved A_∞ -algebras (V, D) and (V', D') is a coalgebra map $F : T(sV) \rightarrow T(sV')$ which is a chain map, i.e. $FD = D'F$.

Proposition 2.4. (cf. [6],[10]) *There is a bijection between morphism of curved A_∞ -algebras $F : (V, \{m_k^V\}) \rightarrow (V', \{m_k^{V'}\})$ and collections of linear maps $\{F_k\}_{k=1}^\infty$ with $F_k : V^{\otimes k} \rightarrow V'$ of degree $1 - k$, satisfying*

$$F_1 m_0^V = m_0^{V'},$$

and for every $n > 0$ the identity

$$\begin{aligned} & \sum_{r+s+t=n} (-1)^{r+st} F_{r+t+1} (\mathbf{1}^{\otimes r} \otimes m_s^V \otimes \mathbf{1}^{\otimes t}) \\ &= \sum_{\substack{1 \leq q \leq n \\ i_1 + \dots + i_q = n}} (-1)^w m_q^{V'} (F_{i_1} \otimes \dots \otimes F_{i_q}), \end{aligned}$$

where

$$w = \sum_{2 \leq \ell \leq q} \left((1 - i_\ell) \sum_{1 \leq k \leq \ell-1} i_k \right).$$

The relationship between a morphism F and its components $\{F_k\}$ is obtained as follows. Setting

$$F_{n,k} := pr_{(sV')^{\otimes k} F} \big|_{(sV)^{\otimes n}} : (sV)^{\otimes n} \rightarrow (sV')^{\otimes k},$$

one has

$$F_{n,k} = \sum_{i_1 + \dots + i_k = n} F_{i_1,1} \otimes \dots \otimes F_{i_k,1},$$

where $F_{k,1}$ coincides with F_k up to a sign obtained by the Koszul sign rule. The components of the composition of two morphisms $F = \{F_k\}_{k=1}^\infty$ and $G = \{G_k\}_{k=1}^\infty$ are given by the formula (cf. [10, Section 4])

$$(2.6) \quad (GF)_p = \sum_{\substack{1 \leq q \leq p \\ i_1 + \dots + i_q = p}} (-1)^w G_q (F_{i_1} \otimes \dots \otimes F_{i_q}).$$

Example 2.5. (Gauge transformations in a dg algebra) Let A be a unital dg algebra with differential d . Every $\gamma \in A$ of degree 1 defines a curved dg structure D_γ on A given by

$$m_0^\gamma(1) = d\gamma + \gamma^2, \quad m_1^\gamma(a) = da + [\gamma, a], \quad m_2^\gamma(a \otimes b) = ab, \quad a, b \in A.$$

We denote by G_A the multiplicative group of all degree 0 invertible elements of A . For every $g \in G_A$ and $a \in A$ we set $c(g)(a) = gag^{-1}$ and $\gamma' = g\gamma g^{-1} + gdg^{-1}$. One easily checks that every $g \in G_A$ defines a morphism F^g from (A, D_γ) to $(A, D_{\gamma'})$ whose components are given by

$$F_1^g = c(g), \quad F_k^g = 0, k > 1,$$

and, using (2.6), that the assignment $g \mapsto F^g$ defines a representation of G_A on $T(sA)$.

3. ALGEBRAIC GAUGE THEORY

We denote by $\text{Aut}(T(V))$ the space of the coalgebra automorphisms of $T(V)$ and by \mathfrak{A}_V the set of all curved dg structures on V . In what follows, by inner product we mean a symmetric non-degenerate bilinear form, and by unitary operator, an invertible inner product preserving map. A *graded* inner product on a graded vector space V is an inner product on V such that $V_i \perp V_j$ for $i \neq j$.

Definition 3.1. Let G be a group and let V be a graded vector space. An *algebraic gauge theory* over G with target V is a triple $(\mathfrak{C}, \langle \cdot, \cdot \rangle, \rho)$, where

- (1) \mathfrak{C} is a subset of \mathfrak{A}_V .
- (2) $\langle \cdot, \cdot \rangle$ is a graded inner product on $T(sV)$,
- (3) $\rho : G \rightarrow \text{Aut}(T(sV))$ is a unitary representation of G on $T(sV)$ via coalgebra maps such that for every $g \in G$ and every $D \in \mathfrak{C}$ one has $\rho(g)D\rho(g^{-1}) \in \mathfrak{C}$ and $\rho(g)1 = 1$.

Motivated by Example 2.5 and following the standard physics terminology, we shall call the elements of $\rho(G)$ *gauge transformations* and the map $S : \mathfrak{C} \rightarrow \mathbf{k}$ given by

$$S(D) = \langle D1, D1 \rangle, \quad D \in \mathfrak{C}$$

the *action functional* of the theory.

Proposition 3.2. *The action functional S is invariant under gauge transformations.*

Proof. For every $g \in G$ one has

$$\begin{aligned} S(\rho(g)D\rho(g^{-1})) &= \langle \rho(g)D\rho(g^{-1})1, \rho(g)D\rho(g^{-1})1 \rangle = \\ &= \langle \rho(g)D1, \rho(g)D1 \rangle = \langle D1, D1 \rangle = S(D). \end{aligned}$$

□

In the remaining part of this section we shall present three examples of algebraic gauge theory. We begin with the following simple observation.

Lemma 3.3. *Let $\alpha : G \rightarrow \text{Aut}(V)$ be a representation of a group G on a inner product vector space V . Assume that the adjoint operator $\alpha^*(g)$ exists for every $g \in G$ and that the commutator $[\alpha(g_1), \alpha^*(g_2)]$ is equal to 0 for all $g_1, g_2 \in G$. Then $\rho = \alpha(\alpha^*)^{-1}$ is a unitary representation of G .*

Proof. We check that ρ is a representation

$$\begin{aligned} \rho(g_1g_2) &= \alpha(g_1g_2)\alpha^*((g_1g_2)^{-1}) = \alpha(g_1)\alpha(g_2)\alpha^*(g_1^{-1})\alpha^*(g_2^{-1}) = \\ &= \alpha(g_1)\alpha^*(g_1^{-1})\alpha(g_2)\alpha^*(g_2^{-1}) = \rho(g_1)\rho(g_2), \end{aligned}$$

and that ρ is unitary

$$\rho^* = ((\alpha^*)^{-1})^* = \alpha^{-1}\alpha^* = \alpha^*\alpha^{-1} = \rho^{-1}.$$

□

3.1. Classical gauge theory. Our first example is classical gauge theory. Let E be a real or complex smooth vector bundle over a Riemannian manifold M . Denote by $\Omega_c^\bullet(M, E)$ the algebra of compactly supported differential forms on M with values in E , by $\Omega_E := \Omega_c^\bullet(M, \text{End}(E))$ the algebra of compactly supported forms with values in the endomorphism bundle $\text{End}(E)$ and by \mathfrak{A}_E the set of all connections on E . For every

$$\mathfrak{A}_E \ni \nabla : \Omega_c^\bullet(M, E) \rightarrow \Omega_c^{\bullet+1}(M, E)$$

one can define the induced connection

$$\tilde{\nabla} : \Omega_c^\bullet(M, \text{End}(E)) \rightarrow \Omega_c^{\bullet+1}(M, \text{End}(E))$$

on $\text{End}(E)$.

We define a curved dg structure $D_\nabla = (m_0^\nabla, m_1^\nabla, m_2^\nabla)$ on Ω_E by setting

$$m_0^\nabla = \nabla^2, \quad m_1^\nabla = \tilde{\nabla}$$

and taking m_2^∇ to be the composition product on Ω_E . Using the Riemannian structure on M , we define an inner product on Ω_E by

$$\langle \omega_1, \omega_2 \rangle_E = \int_M \text{Tr}(\omega_1 \wedge * \omega_2), \quad \omega_1, \omega_2 \in \Omega_E.$$

The latter induces a graded inner product on the tensor coalgebra $T(s\Omega)$ which we also denote by $\langle \cdot, \cdot \rangle_E$.

Let G_E be a subgroup of the sections of the automorphism bundle $\text{Aut}(E)$. We denote by $\rho : G_E \rightarrow \text{Aut}(\Omega_E)$ the representation of G_E on Ω_E given by conjugation. The latter induces a representation $\rho_E : G_E \rightarrow \text{Aut}(T(s\Omega_E))$.

Proposition 3.4. *The triple $(\{D_\nabla\}_{\nabla \in \mathfrak{A}_E}, \langle \cdot, \cdot \rangle_E, \rho_E)$ defines an algebraic gauge theory over G_E with target Ω_E .*

Proof. We observe that the operators of left multiplication and right multiplication by an element of G_E are adjoint and that the representation ρ is given by the product of the left multiplication and the inverse of the right multiplication. Hence ρ is unitary by Lemma 3.3 and the same is true for ρ_E . For $g \in G_E$ we denote by ∇^g the usual action of G_E on connections. A straightforward computation shows that $\rho_E(g)D_\nabla\rho_E(g^{-1}) = D_{\nabla^g}$. \square

3.2. Spectral triples. Our second example is based on A. Connes' spectral triples [1]. Let \mathcal{A} be an associative algebra and let \mathcal{M} be a finitely generated projective left \mathcal{A} -module. Let Ω^\bullet be a differential graded algebra such that Ω^0 is isomorphic to \mathcal{A} ; thus Ω^\bullet is an \mathcal{A} -bimodule. Using this data, one defines connections (and their curvatures) on \mathcal{M} in the standard fashion (see e.g. [3].) More specifically, a *connection* on the pair $(\mathcal{M}, \Omega^\bullet)$ is a degree 1 map

$$\nabla : \Omega^\bullet \otimes_{\mathcal{A}} \mathcal{M} \rightarrow \Omega^\bullet \otimes_{\mathcal{A}} \mathcal{M}$$

that satisfies the graded Leibniz rule. As in the commutative case, a connection ∇ on $(\mathcal{M}, \Omega^\bullet)$ induces a connection on $(\text{End}_{\mathcal{A}}(\mathcal{M}), \Omega^\bullet)$ and hence a curved dg structure D_∇ on $\Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M})$.

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd spectral triple. Thus \mathcal{A} is a unital complex algebra with involution faithfully represented on a Hilbert space \mathcal{H} via an involution preserving map $\phi : \mathcal{A} \rightarrow \text{End}(\mathcal{H})$, where $\text{End}(\mathcal{H})$ denotes the algebra of bounded operators on \mathcal{H} , and \mathcal{D} is an unbounded densely defined self-adjoint operator on \mathcal{H} with compact resolvent such that $[\mathcal{D}, \phi(a)]$ is bounded for every $a \in \mathcal{A}$. We suppose that $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ satisfies the summability and regularity conditions stated in [2] so that the algebra of pseudodifferential operators $\Psi^\bullet(\mathcal{A})$ associated to $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and a residue trace $\tau_{\mathcal{A}}$ on $\Psi^\bullet(\mathcal{A})$ exist (cf. [2, Proposition II.1]).

We assume without loss of generality that \mathcal{D} is invertible, set $F = \mathcal{D}|\mathcal{D}|^{-1}$ and denote by $\Omega^\bullet(\mathcal{H}, F)$ the dg algebra generated by the Fredholm module (\mathcal{H}, F) (cf. [1, IV.1.α]). By construction $\Omega^\bullet(\mathcal{H}, F)$ is a subalgebra of $\text{End}(\mathcal{H})$ and $\Omega^0(\mathcal{H}, F)$ is isomorphic to \mathcal{A} . We further suppose that $\Omega^\bullet(\mathcal{H}, F) \subset \Psi^\bullet(\mathcal{A})$ and that the symmetric bilinear form defined by $\tau_{\mathcal{A}}$ is non-degenerate on $\Omega^\bullet(\mathcal{H}, F)$.

Let $\text{Tr}_{\mathcal{M}} : \text{End}_{\mathcal{A}}(\mathcal{M}) \rightarrow \mathcal{A}$ be the matrix trace and set

$$\tau_{\mathcal{M}}(\omega \otimes m) = \tau_{\mathcal{A}}(\omega \otimes \text{Tr}_{\mathcal{M}} m), \quad \omega \otimes m \in \Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M}).$$

One easily checks that the latter defines a trace on $\Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M})$ and that

$$\langle x, y \rangle_{\mathcal{D}} = \tau_{\mathcal{M}}(xy), \quad x, y \in \Omega^k \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M})$$

defines an inner product on $\Omega^k \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M})$ and hence a graded inner product on $T(s\Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M}))$.

Let $G_{\mathcal{M}}$ be a subgroup of $\text{Aut}_{\mathcal{A}}(\mathcal{M})$. The representation of $G_{\mathcal{M}}$ on $\text{End}_{\mathcal{A}}(\mathcal{M})$ given by conjugation extends to a representation on $\Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M})$ and hence

also to a representation $\rho_{\mathcal{M}}$ on $T(s\Omega^\bullet \otimes_{\mathcal{A}} \text{End}_{\mathcal{A}}(\mathcal{M}))$. Let $\mathfrak{A}_{\mathcal{M}}$ be the set of all connections on the pair $(\mathcal{M}, \Omega^\bullet(\mathcal{H}, F))$.

Proposition 3.5. *The triple $(\{D_\nabla\}_{\nabla \in \mathfrak{A}_{\mathcal{M}}}, \langle \cdot, \cdot \rangle_{\mathcal{M}}, \rho_{\mathcal{M}})$ defines an algebraic gauge theory over $G_{\mathcal{M}}$ with target $\Omega^\bullet \otimes \text{End}_{\mathcal{A}}(\mathcal{M})$.*

Proof. The proof is completely analogous to that of Proposition 3.4. \square

3.3. Deformations of dg algebras. Our third example is based on curved dg structures that are deformations. Let A be a unital dg algebra. Recall that by Example 2.5 every degree 1 element γ of A defines a curved dg structure D_γ on A . Let G_A be a subgroup of the group of all invertible degree 0 elements of A . Assume that A is equipped with a graded inner product such that action of G_A on A given by right multiplication is adjoint to the action given by left multiplication. This inner product induces an inner product $\langle \cdot, \cdot \rangle$ on $T(sA)$. The action of G_A on A by conjugation induces a representation ρ of G_A on $T(sA)$ as already observed in Example 2.5.

Proposition 3.6. *The triple $(\{D_\gamma\}_{\gamma \in A_1}, \langle \cdot, \cdot \rangle, \rho)$ defines an algebraic gauge theory over G_A with target A .*

Proof. The proof is analogous to that of Proposition 3.4. The fact that $\rho(g)D_\gamma = D_{\gamma'}\rho(g)$ for every $g \in G_A$ and $\gamma' = g\gamma g^{-1} + gdg^{-1}$ was already observed in Example 2.5. \square

4. SPECIAL CHAIN CONTRACTIONS AND THE PERTURBATION LEMMA

4.1. Special chain contractions. In this subsection we discuss certain properties of chain maps transferred via special chain contractions.

Definition 4.1. Assume that we are given two chain complexes (C, d) and (C', d') and chain maps $p : C \rightarrow C'$ and $i : C' \rightarrow C$. We say that the pair (p, i) is a *chain contraction* of (C, d) to (C', d') if $pi = 1_{C'}$ and there exists a homotopy between ip and the identity map on C , i.e. a map $H : C^\bullet \rightarrow C^{\bullet-1}$ such that

$$(4.1) \quad ip - 1_C = dH + Hd.$$

We say that the contraction (p, i, H) is *special* if the following so called annihilation conditions hold.

$$(4.2) \quad Hi = 0, \quad pH = 0, \quad H^2 = 0.$$

It turns out that every chain contraction may be modified to a special one [9]. In the next three lemmas we assume that we are given special chain contractions (p_k, i_k, H_k) of (C_k, d_k) to (C'_k, d'_k) , $k = 1, 2, 3$. Given a chain map $\phi : C_k \rightarrow C_l$, we set

$$\widehat{\phi} = p_l \phi i_k.$$

Lemma 4.2. *Let $\phi : C_1 \rightarrow C_2$ and $\psi : C_2 \rightarrow C_3$ be chain maps such that $\phi H_1 = H_2 \phi$ and $\psi H_2 = H_3 \psi$. Then*

$$\widehat{\psi \phi} = \widehat{\psi} \widehat{\phi}.$$

Proof. We compute using (4.1) and (4.2) as follows:

$$\begin{aligned} \widehat{\psi \phi} &= p_3 \psi i_2 p_2 \phi i_1 = p_3 \psi (1 + d_2 H_2 + H_2 d_2) \phi i_1 = \\ &= p_3 \psi \phi i_1 + p_3 \psi d_2 \phi H_1 i_1 + p_3 H_3 \psi d_2 \phi i_1 = \widehat{\psi} \widehat{\phi}. \end{aligned}$$

\square

Lemma 4.3. *Let $\langle \cdot, \cdot \rangle_k$ be an inner product on C_k , $k = 1, 2$. Let $\phi : C_1 \rightarrow C_2$ and $\psi : C_2 \rightarrow C_1$ be chain maps satisfying $\phi H_1 = H_2 \phi$ and $\psi H_2 = H_1 \psi$ and such that ψ is the adjoint of ϕ . Then $\widehat{\psi}$ is the adjoint of $\widehat{\phi}$ with respect to the inner products $\langle i_k \cdot, i_k \cdot \rangle_k$ on C'_k , $k = 1, 2$.*

Proof. For every $c_k \in C'_k$, $k = 1, 2$ one has, using (4.1) and (4.2)

$$\begin{aligned} \langle i_2 \widehat{\phi} c_1, i_2 c_2 \rangle_2 &= \langle i_2 p_2 \phi i_1 c_1, i_2 c_2 \rangle_2 = \langle (1 + d_2 H_2 + H_2 d_2) \phi i_1 c_1, i_2 c_2 \rangle_2 = \\ &= \langle \phi i_1 c_1, i_2 c_2 \rangle_2 + \langle d_2 \phi H_1 i_1 c_1, i_2 c_2 \rangle_2 + \langle \phi H_1 i_1 d'_1 c_1, i_2 c_2 \rangle_2 = \langle \phi i_1 c_1, i_2 c_2 \rangle_2. \end{aligned}$$

On the other hand, one has

$$\begin{aligned} \langle i_1 c_1, i_1 \widehat{\psi} c_2 \rangle_1 &= \langle i_1 c_1, i_1 p_1 \psi i_2 c_2 \rangle_1 = \langle i_1 c_1, (1 + d_1 H_1 + H_1 d_1) \psi i_2 c_2 \rangle_1 = \\ &= \langle i_1 c_1, \psi i_2 c_2 \rangle_1 + \langle i_1 c_1, d_1 \psi H_2 i_2 c_2 \rangle_1 + \langle i_1 c_1, \psi H_2 i_2 d'_2 c_2 \rangle_1 = \langle i_1 c_1, \psi i_2 c_2 \rangle_1. \end{aligned}$$

□

Lemma 4.4. *Let $\langle \cdot, \cdot \rangle_k$ be an inner product on C_k , $k = 1, 2$ and let $\phi : C_1 \rightarrow C_2$ be a unitary chain map satisfying $\phi H_1 = H_2 \phi$. Then $\widehat{\phi}$ is unitary with respect to the inner products $\langle i_k \cdot, i_k \cdot \rangle_k$ on C'_k , $k = 1, 2$.*

Proof. This follows directly from Lemmas 4.2 and 4.3. □

4.2. The perturbation lemma. We recall the coalgebra homological perturbation lemma from [7] as stated in [11]. Given a chain complex (C, d) , we say that a map $\delta : C^\bullet \rightarrow C^{\bullet+1}$ is a *perturbation* of d if $(d + \delta)^2 = 0$.

Theorem 4.5. (cf. [7, Section 2]) (a) *Let (C_1, d_1) be a chain complex and δ_1 be a perturbation of d_1 . Suppose that we are given a special chain contraction (p, i, H) between (C_1, d_1) and a chain complex (C_2, d_2) . Assume further that $\mathbf{1}_{C_1} - \delta_1 H$ is invertible and set $\Sigma = (\mathbf{1}_{C_1} - \delta_1 H)^{-1} \delta_1$. Then $\delta_2 = p \Sigma i$ is a perturbation of d_2 and the formulas*

$$\widetilde{i} = i + H \Sigma i, \quad \widetilde{p} = p + p \Sigma H, \quad \widetilde{H} = H + H \Sigma H$$

define a special chain contraction between $(C_1, d_1 + \delta_1)$ and $(C_2, d_2 + \delta_2)$.

(b) *Assume, in addition to the hypotheses in part (a), that (C_1, d_1) is dg coalgebra, δ_1 is a coderivation, and p and i are coalgebra homomorphisms. Then δ_2 is a coderivation and the maps \widetilde{p} and \widetilde{i} are coalgebra homomorphisms.*

We shall recall how this theorem may be used to transfer curved A_∞ -structures and morphisms in the next section.

Lemma 4.6. *Assume that we are given two copies of the data described in Theorem 4.5, i.e. a special chain contraction (p, i, H) of (C_1, d_1) to (C_2, d_2) , a special chain contraction (p', i', H') of (C'_1, d'_1) to (C_2, d'_2) , a perturbation δ_1 of d_1 and a perturbation δ'_1 of d'_1 . Let $\varphi : (C_1, d_1 + \delta_1) \rightarrow (C'_1, d'_1 + \delta'_1)$ be a chain map such that*

$$(4.3) \quad \varphi H = H' \varphi.$$

Then $\varphi \widetilde{H} = \widetilde{H}' \varphi$.

Proof. We need to show that

$$(4.4) \quad \varphi H \Sigma H = H' \Sigma' H' \varphi.$$

We can write

$$\Sigma = 1 + \sum_{n=1}^{\infty} (\delta_1 H)^n \delta_1$$

which using $H^2 = 0$ implies

$$(4.5) \quad H \Sigma H = \sum_{n=2}^{\infty} (H \delta_1)^n H$$

Now it follows from (4.5) and (4.3) that in order to prove (4.4) it suffices to show that

$$(4.6) \quad \varphi H \delta_1 H = H' \delta'_1 H' \varphi.$$

Using that $\varphi(d_1 + \delta_1) = (d'_1 + \delta'_1)\varphi$ we see that it suffices to prove that

$$(4.7) \quad H'(\varphi d_1 - d'_1 \varphi)H = 0$$

Finally, we verify (4.7) using (4.2) and (4.3):

$$\begin{aligned} H'(\varphi d_1 - d'_1 \varphi)H &= H' \varphi(ip - 1 - Hd_1) - (i'p' - 1 - H'd'_1)\varphi H = \\ &= H' \varphi ip - H' \varphi - H' \varphi Hd_1 - i'p' \varphi H + \varphi H + d'_1 H' \varphi H = 0. \end{aligned}$$

□

5. ALGEBRAIC GAUGE THEORY OVER A GROUPOID

5.1. Definition and main properties. In this section we generalize the definition of algebraic gauge theory from Section 3 by replacing the curved dg structures with curved A_∞ -structures and the group by a groupoid. We denote the set of all curved curved A_∞ -structures on V by \mathfrak{B}_V .

Definition 5.1. Let \mathbf{J} be the category whose objects are all triples $(V, \langle \cdot, \cdot \rangle_{T(sV)}, \mathfrak{C}_V)$, where V is a graded vector space, $\langle \cdot, \cdot \rangle_{T(sV)}$ is a graded inner product on $T(sV)$ and \mathfrak{C}_V is a subset of \mathfrak{B}_V , and whose morphisms are pairs (ϕ, ψ) of unit-preserving partially isometric coalgebra morphisms

$$\phi : T(sV_1) \rightarrow T(sV_2), \quad \psi : T(sV_2) \rightarrow T(sV_1)$$

such that

$$\phi \mathfrak{C}_{V_1} \psi \subset \mathfrak{C}_{V_2}$$

with the obvious composition law.

Definition 5.2. Let \mathbf{G} be a groupoid. An *algebraic gauge theory* over \mathbf{G} is a functor $\mathcal{F} : \mathbf{G} \rightarrow \mathbf{J}$.

If \mathbf{G} is a groupoid, we denote the set of units of \mathbf{G} by $\mathbf{G}^{(0)}$ and the source and target maps by $s, t : \mathbf{G} \rightarrow \mathbf{G}^{(0)}$. We denote the image of $g \in \mathbf{G}^{(0)}$ under \mathcal{F} by $(V_g, \langle \cdot, \cdot \rangle_g, \mathfrak{C}_g)$. We call the morphisms in the image of \mathcal{F} *gauge transformations* and the collection of functionals $\{S_g\}_{g \in \mathbf{G}^{(0)}}$ given by

$$S_g(D) = \langle D1, D1 \rangle_g, \quad D \in \{\mathfrak{C}_g\}_{g \in \mathbf{G}^{(0)}}$$

the *action functional* of the theory. As in Section 3, one shows that the action functional is invariant under gauge transformations, i.e. for every $g \in \mathbf{G}$ and every $D \in \mathfrak{C}_{s(g)}$ one has

$$S_{t(g)}(\mathcal{F}(g)D\mathcal{F}^{-1}(g)) = S_{s(g)}(D).$$

In this paper we discuss only examples of algebraic gauge theory over \mathbf{G} in which all spaces $\{V_g\}_{g \in \mathbf{G}^{(0)}}$ coincide with a fixed space V . In this special case the representation of the groupoid \mathbf{G} given by \mathcal{F} defines an equivalence relation on the collection of curved A_∞ -structures on V .

5.2. Example: Transferred curved A_∞ -structures. Let A be a unital real or complex dg algebra and G be a subgroup of the group of all invertible degree 0 elements of A . We consider a triple $(\{D_\gamma\}_{\gamma \in A_1}, \langle \cdot, \cdot \rangle, \rho)$ as in Proposition 3.6.

Let (p, i, H) be a special contraction from A to a chain complex B and assume that the conditions stated in [11, Proposition 3.3] hold with respect to a fixed norm on A and the norm on B induced by the inclusion i . Thus according to [11, Proposition 3.3] there exists a subset Γ_A of A_1 such that for each $\gamma \in \Gamma_A$ the curved dg structure D_γ on A may be transferred to a curved A_∞ -structure \tilde{D}_γ on B . Moreover, by the homological perturbation lemma, for every $\gamma \in \Gamma_A$ the special contraction (p, i, H) induces a special contraction $(P_\gamma, I_\gamma, H_\gamma)$ from $T(sA)$ to $T(sB)$ such that P_γ and I_γ are morphisms of curved A_∞ -algebras. Thus the injection I_γ induces an inner product on $T(sB)$ which we denote by $\langle \cdot, \cdot \rangle_{B, \gamma}$.

We define a groupoid \mathbf{G}_A by setting

$$\mathbf{G}_A = \{(\gamma, g, \gamma') \in \Gamma_A \times G \times \Gamma_A \mid g \cdot \gamma = \gamma'\},$$

$$\mathbf{G}_A^{(0)} = \Gamma_A, \quad s(\gamma, g, \gamma') = \gamma, \quad t(\gamma, g, \gamma') = \gamma',$$

and using the obvious composition law given by the multiplication in G . Exactly as in Subsection 3.3 one verifies the following

Proposition 5.3. *The assignment*

$$\Gamma_A \ni \gamma \mapsto (A, \langle \cdot, \cdot \rangle, \{D_\gamma\}_{\gamma \in \mathbf{G}_A^{(0)}}),$$

$$\mathbf{G}_A \ni (\gamma, g, \gamma') \mapsto \rho(g)$$

defines an algebraic gauge theory $\mathcal{F}_A : \mathbf{G}_A \rightarrow \mathbf{J}$.

Next we show how this algebraic gauge theory can be transferred along the contraction (p, i, H) . For every $(\gamma, g, \gamma') \in \mathbf{G}_A$ one can transfer the morphism of curved dg algebras

$$\rho(g) : (A, D_\gamma) \rightarrow (A, D_{\gamma'})$$

to a morphism of curved A_∞ -algebras

$$\tilde{\rho}_{\gamma, \gamma'}(g) := P_{\gamma'} \rho(g) I_\gamma : (B, \tilde{D}_\gamma) \rightarrow (B, \tilde{D}_{\gamma'}).$$

Theorem 5.4. *Assume that for every $g \in G$ one has $[H, c(g)] = 0$, where $c(g)$ denotes conjugation with g . Then the assignment*

$$\Gamma_A \ni \gamma \mapsto (B, \langle \cdot, \cdot \rangle_{B, \gamma}, \{\tilde{D}_\gamma\}_{\gamma \in \mathbf{G}_A^{(0)}}),$$

$$\mathbf{G}_A \ni (\gamma, g, \gamma') \mapsto \tilde{\rho}_{\gamma, \gamma'}(g)$$

defines an algebraic gauge theory $\mathcal{F}_B : \mathbf{G}_A \rightarrow \mathbf{J}$. Moreover, there exists a natural transformation from \mathcal{F}_A to \mathcal{F}_B given by the assignment

$$\mathbf{G}_A^{(0)} \ni \gamma \mapsto (P_\gamma, I_\gamma) \in \text{Hom}(\mathcal{F}_A(\gamma), \mathcal{F}_B(\gamma)).$$

Proof. Recall that the special contraction (p, i, H) induces a special contraction $(T(p), T(i), T(H))$ from $T(sA)$ to $T(sB)$. The condition $[H, c(g)] = 0$ implies that $[T(H), \rho(g)] = 0$. Applying Lemma 4.6 to the maps $\rho(g)$, one obtains

$$H_{\gamma'} \rho(g) = \rho(g) H_\gamma$$

for $\gamma' = g \cdot \gamma$. Using this, Theorem 4.5 and the unitarity of $\rho(g)$, one concludes that by virtue of Lemma 4.4 the operators $\tilde{\rho}_{\gamma, \gamma'}(g)$ are isometries. Lemma 4.2 implies that

$$\tilde{\rho}_{\gamma', \gamma''}(g_2) \tilde{\rho}_{\gamma, \gamma'}(g_1) = \tilde{\rho}_{\gamma, \gamma''}(g_2 g_1)$$

for $\gamma' = g_1 \cdot \gamma$ and $\gamma'' = g_2 \cdot \gamma'$. Thus the maps $\tilde{\rho}_{\gamma, \gamma'}(g)$ define a representation of the groupoid \mathbf{G}_A on the collection $\{B\}_{\gamma \in \mathbf{G}_A^{(0)}}$ and hence a representation on $\{T(sB)\}_{\gamma \in \mathbf{G}_A^{(0)}}$. This implies that $\tilde{\rho}_{\gamma, \gamma'}(g)$ are isometric isomorphisms.

To prove that the assignment

$$\mathbf{G}_A^{(0)} \ni \gamma \mapsto (P_\gamma, I_\gamma) \in \text{Hom}(\mathcal{F}_A(\gamma), \mathcal{F}_B(\gamma))$$

defines a natural transformation from \mathcal{F}_A to \mathcal{F}_B we first observe that

$$P_\gamma D_\gamma I_\gamma = \tilde{D}_\gamma$$

and then show that the identities

$$\begin{aligned} P_{\gamma'} \rho(g) &= \tilde{\rho}_{\gamma, \gamma'}(g) P_\gamma, \\ \rho(g) I_\gamma &= I_{\gamma'} \tilde{\rho}_{\gamma, \gamma'}(g), \\ P_{\gamma'} \rho(g) D_{\gamma'} \rho^{-1}(g) I_{\gamma'} &= \tilde{\rho}_{\gamma, \gamma'}(g) P_\gamma D_\gamma I_\gamma \tilde{\rho}_{\gamma, \gamma'}^{-1}(g) \end{aligned}$$

hold for $\gamma' = g \cdot \gamma$. These are verified as above, using the annihilation conditions and the identities $\tilde{D}_\gamma P_\gamma = P_\gamma D_\gamma$ and $I_\gamma \tilde{D}_\gamma = D_\gamma I_\gamma$. \square

5.3. Example: A simplicial gauge theory. Here we consider a special case of the example from the previous subsection in order to define a discretization of classical gauge theory. Let M be a compact Riemannian manifold and let K be a smooth triangulation of M . We denote by $C^\bullet(K)$ the space of the real or complex simplicial cochains on K and by $\Omega^\bullet(K)$ the space of the real or complex piece-wise smooth differential forms on K (cf. [4]). We further denote by \mathbf{M}_l the algebra of the real or complex $l \times l$ -matrices and by $C^\bullet(K, \mathbf{M}_l)$ and $\Omega^\bullet(K, \mathbf{M}_l)$ the spaces of matrix-valued cochains and forms.

Recall that there exists a chain contraction from $\Omega^\bullet(K)$ to $C^\bullet(K)$ (cf. [4, Theorem 2.16] and that it is proved in [5, Section 3] that this contraction is special. Clearly this contraction extends to a contraction from $\Omega^\bullet(K, \mathbf{M}_l)$ to $C^\bullet(K, \mathbf{M}_l)$.

We define an L^2 -inner product on $\Omega^\bullet(K, \mathbf{M}_l)$ as in Subsection 3.1, fix $e > 0$ and set

$$\Gamma_e = \{\gamma \in \Omega^1(K, \mathbf{M}_l) \mid \|\gamma\| \leq e\},$$

where $\|\cdot\|$ is the norm induced by the inner product. It is shown in [11, Example 3.9] that there is a fine enough subdivision K_e of K such that every $\gamma \in \Gamma_e$ defines a transferred curved A_∞ -structure D_γ^e on $C^\bullet(K_e, \mathbf{M}_l)$ as in subsection 4.1, using Dupont's contraction.

Let G be a subgroup of the group of all invertible elements of $\Omega^0(K, \mathbf{M}_l)$. We define a family of graded inner products $\langle \cdot, \cdot \rangle_{e, \gamma}$ on $T(sC^\bullet(K_e, \mathbf{M}_l))$, a groupoid \mathbf{G}_e with unit space Γ_e and a representation $\rho^e = \{\rho_{\gamma, \gamma'}^e(g)\}_{(\gamma, g, \gamma') \in \mathbf{G}_e}$ of \mathbf{G}_e on $T(sC^\bullet(K_e, \mathbf{M}_l))$ exactly as in subsection 4.1.

As in Proposition 5.3, we obtain an algebraic gauge theory $\mathcal{F}_\Omega : \mathbf{G}_e \rightarrow \mathbf{J}$ given by the assignment

$$\begin{aligned} \Gamma_e \ni \gamma &\mapsto (\Omega^\bullet(K, \mathbf{M}_l), \langle \cdot, \cdot \rangle, \{D_\gamma\}_{\gamma \in \Gamma_e}), \\ \mathbf{G}_e \ni (\gamma, g, \gamma') &\mapsto \rho(g). \end{aligned}$$

Theorem 5.5. *The assignment*

$$\begin{aligned} \Gamma_e \ni \gamma &\mapsto (C^\bullet(K_e, \mathbf{M}_l), \langle \cdot, \cdot \rangle_{e, \gamma}, \{D_\gamma^e\}_{\gamma \in \Gamma_e}), \\ \mathbf{G}_e \ni (\gamma, g, \gamma') &\mapsto \rho_{\gamma, \gamma'}^e(g) \end{aligned}$$

defines an algebraic gauge theory $\mathcal{F}_C : \mathbf{G}_e \rightarrow \mathbf{J}$. Moreover, there exists a natural transformation from \mathcal{F}_Ω to \mathcal{F}_C .

Proof. It follows directly from the definition of the homotopy in Dupont's contraction given in [4, Chapter 2] that it commutes with multiplication by 0-forms, hence it commutes with conjugations by elements of G and Theorem 5.4 is applicable. \square

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